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An Isserlis' Theorem for Mixed Gaussian Variables: Application to the Auto-Bispectral Density

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Abstract This work derives a version of Isserlis' theorem for the specific case of four mixed-Gaussian random variables. The theorem is then used to derive an expression for the auto-bispectral density for quadratically nonlinear systems driven with mixed-Gaussian iid noise.

Keywords Isserlis' theorem · Wick's theorem · Mixed-Gaussian distribution · Auto-bispectral density

1 Introduction

In the study of random processes, Isserlis' theorem can be used to relate the expectation of products of even numbers of normalized, jointly Gaussian random variables to the sum of averages of products of pairs of these variables. More specifically,

Theorem 1.1 (Isserlis' Theorem) *If* $\eta_1, \eta_2, \ldots, \eta_{2N+1}$ $(N = 1, 2, \ldots)$ *are normalized, jointly Gaussian random variables* (*i.e., for every* i, $E[\eta_i] = 0$ *and* $E[\eta_i^2] = 1$), *then*

$$E[\eta_1\eta_2\cdots\eta_{2N}] = \sum\prod E[\eta_i\eta_j]$$

and

$$E[\eta_1\eta_2\cdots\eta_{2N+1}] = 0$$

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where the notation $\sum\prod$ means summing over all distinct ways of partitioning $\eta_1, \eta_2, \dots, \eta_{2N}$ into pairs. For example, consider the case of four Gaussian random variables $\eta_1, \eta_2, \eta_3, \eta_4$. Application of the above theorem gives the relation

$$E[\eta_1\eta_2\eta_3\eta_4] = E[\eta_1\eta_2]E[\eta_3\eta_4] + E[\eta_1\eta_3]E[\eta_2\eta_4] + E[\eta_1\eta_4]E[\eta_2\eta_3]. \tag{1}$$

If the variables are not normalized but are still jointly Gaussian; that is to say $\xi_1, \xi_2, \dots, \xi_{2N+1}$ are jointly Gaussian random variables with $E[\xi_i] = \mu_i$ and $E[(\xi_i - \mu_i)^2] = \sigma_i^2$, the above formulas still apply to give

$$E\left[\left(\frac{\xi_1-\mu_1}{\sigma_1}\right)\left(\frac{\xi_2-\mu_2}{\sigma_2}\right)\cdots\left(\frac{\xi_{2N}-\mu_{2N}}{\sigma_{2N}}\right)\right] = \sum\prod E\left[\left(\frac{\xi_i-\mu_i}{\sigma_i}\right)\left(\frac{\xi_j-\mu_j}{\sigma_j}\right)\right] \tag{2}$$

$$E\left[\left(\frac{\xi_1-\mu_1}{\sigma_1}\right)\left(\frac{\xi_2-\mu_2}{\sigma_2}\right)\cdots\left(\frac{\xi_{2N+1}-\mu_{2N+1}}{\sigma_{2N+1}}\right)\right] = 0 \tag{3}$$

and, if you multiply each side by the product $\sigma_1\sigma_2\cdots\sigma_{2N}$ in (2) or by $\sigma_1\sigma_2\cdots\sigma_{2N+1}$ in (3), one obtains:

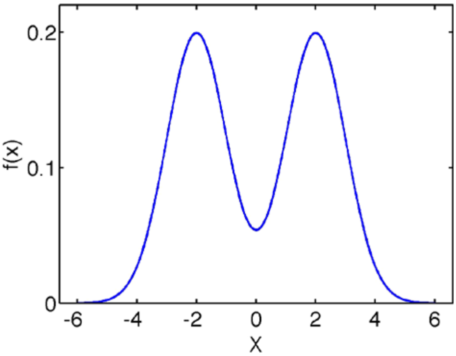
$$E[(\xi_1-\mu_1)(\xi_2-\mu_2)\cdots(\xi_{2N}-\mu_{2N})] = \sum\prod E[(\xi_i-\mu_i)(\xi_j-\mu_j)] \tag{4}$$

$$E[(\xi_1-\mu_1)(\xi_2-\mu_2)\cdots(\xi_{2N+1}-\mu_{2N+1})] = 0 \tag{5}$$

This theorem was first derived by L. Isserlis [6] but is often referred to in the literature as "Wick's theorem" after the work of G.-C. Wick [17] which presented a similar relationship in the context of particle physics. This relationship has proven useful in problems of nonlinear random vibrations [15], the analysis of a portfolio of stock returns [13], quantum field theory [4, 11], and deriving expressions for higher-order spectra associated with a nonlinear system response [7, 9]. This theorem has also been invoked in the development of an algorithm for generating colored noise [1].

Recently there has been some interest in extending the theorem to non-Gaussian distributed random variables. Repetowicz and Richmond [12] have derived a version of the theorem for non-Gaussian variables. Vignat and Bhatnagar have also looked at extending this theorem to a certain class of non-Gaussian distributions [13]. In this work we are also interested in a non-Gaussian Isserlis' theorem, considering the specific case of four jointly mixed-Gaussian distributed random variables. The motivation for the work stems from the aforementioned application of higher-order spectral analysis. The auto-bispectral density function (defined in Sect. 3) is a frequently used tool in the detection of nonlinearity in structural systems [8, 14, 16]. To date, an analytical expression for the auto-bispectral density function of a nonlinear system output exists only if the system is driven with Gaussian noise. However, structures will often be subject to random vibrations from other, possibly non-Gaussian sources. Here we consider a generalization of the Gaussian distribution, the mixed-Gaussian distribution. This particular distribution appears in a number of contexts and provides a useful way to obtain both Gaussian and highly non-Gaussian distributions by varying a single parameter. Here it will be shown that for a fixed input power, driving a nonlinear structure with mixed-Gaussian noise produces bispectral peaks that can be much higher than for Gaussian-driven systems. Thus, when detecting nonlinearity in a structure the shape of the input distribution can significantly impact the detection problem.

Fig. 1 Probability density function of a mixed-Gaussian distribution ($\mu = 2.0,\ \sigma = 1.0$)



2 Isserlis' Theorem for Four Jointly Mixed-Gaussian Random Variables

The derivation of the classic Isserlis' Theorem 1.1 depends on the special form of the characteristic function for normalized jointly Gaussian random variables and is difficult to generalize for other distributions (see Appendix A of [15]). Here we do not directly generalize the classic Isserlis' theorem, but rather use it to develop formulas for the averages of products of variables with the more general mixed-Gaussian or bimodal distribution. A mixed-Gaussian distribution for a single random variable X has a probability density function given by

$$f(x) = \frac{1}{2\sqrt{2\pi}\sigma}\left[e^{-(x-\mu)^2/2\sigma^2} + e^{-(x+\mu)^2/2\sigma^2}\right], \quad -\infty < x < \infty \tag{6}$$

This density function is obtained by splitting a Gaussian density into two parts, centering one half about $+\mu$ and the other about $-\mu$ and summing the resultants. Such a density function is depicted in Fig. 1. The mixed-Gaussian distribution for two variables is formed in the analogous way by summing two bivariate Gaussian distributions giving

$$\begin{aligned} f(x,y) = &\frac{1}{4\pi\sigma_x\sigma_y}\left[\frac{1}{\sqrt{1-\rho_+^2}}e^{-\frac{1}{2(1-\rho_+^2)}\left((\frac{x-\mu_x}{\sigma_x})^2 - 2\rho_+(\frac{x-\mu_x}{\sigma_x})(\frac{y-\mu_y}{\sigma_y}) + (\frac{y-\mu_y}{\sigma_y})^2\right)}\right. \\ &\left. + \frac{1}{\sqrt{1-\rho_-^2}}e^{-\frac{1}{2(1-\rho_-^2)}\left((\frac{x+\mu_x}{\sigma_x})^2 - 2\rho_-(\frac{x+\mu_x}{\sigma_x})(\frac{y+\mu_y}{\sigma_y}) + (\frac{y+\mu_y}{\sigma_y})^2\right)}\right] \end{aligned} \tag{7}$$

where

$$\rho_+ = \frac{E[(x-\mu_x)(y-\mu_y)]}{\sigma_x\sigma_y}, \qquad \rho_- = \frac{E[(x+\mu_x)(y+\mu_y)]}{\sigma_x\sigma_y}$$

That this is the natural extension to two variables is confirmed by checking that the marginal distributions are indeed mixed-Gaussian. In general, the mixed-Gaussian distribution in M variables is obtained by summing two Gaussian distributions in M variables, one centered at $(\mu_1, \mu_2, \ldots, \mu_M)$ and the other at $(-\mu_1, -\mu_2, \ldots, -\mu_M)$ and multiplying the resultant by $\frac{1}{2}$. What enables the result in this paper is the fact that taking any average over a mixed-Gaussian can be obtained by averaging the same expression over the two individual Gaussians, adding the results, and multiplying by $\frac{1}{2}$.

The case of interest to us is the product of four jointly mixed-Gaussian random variables. We prove the following theorem.

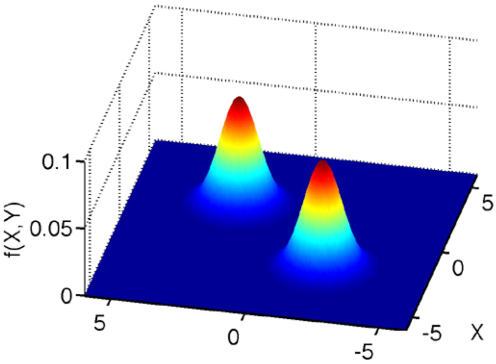


Fig. 2 Probability density function of a joint mixed-Gaussian distribution ($\mu = 2.0,\ \sigma = 1.0$)

Theorem 2.1 *If x_1, x_2, x_3, x_4 are four jointly mixed-Gaussian random variables with parameters μ_i, σ_i for $i = 1, 2, 3, 4$, and $\eta_1, \eta_2, \eta_3, \eta_4$ are four jointly Gaussian random variables with parameters $\mu_i = 0$ and σ_i for $i = 1, 2, 3, 4$, then*

$$\begin{aligned} E[x_1x_2x_3x_4] = {} & E[\eta_1\eta_2]E[\eta_3\eta_4] + E[\eta_1\eta_3]E[\eta_2\eta_4] + E[\eta_1\eta_4]E[\eta_2\eta_3] \\ & + \mu_1\mu_2E[\eta_3\eta_4] + \mu_1\mu_3E[\eta_2\eta_4] + \mu_2\mu_3E[\eta_1\eta_4] + \mu_1\mu_4E[\eta_2\eta_3] \\ & + \mu_2\mu_4E[\eta_1\eta_3] + \mu_3\mu_4E[\eta_1\eta_2] + \mu_1\mu_2\mu_3\mu_4 \end{aligned} \tag{8}$$

In this formula, the average on the left hand side of the equation is over the four-dimensional jointly mixed-Gaussian distribution while the averages on the right-hand side are all over two-dimensional, zero-mean jointly Gaussian distributions with standard deviations σ_i. The first three terms on the right-hand side are the Gaussian result in (1), and our result reduces to the Gaussian case when $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0$.

Proof The average of $x_1x_2x_3x_4$ over the four-dimensional mixed-Gaussian distribution is obtained by averaging over two four-dimensional *Gaussian* distributions and combining the results. Let ξ_i for $i = 1, 2, 3, 4$ denote four jointly Gaussian distributed random variables with parameters μ_i, σ_i for $i = 1, 2, 3, 4$. To find the average over the joint Gaussian centered at $(\mu_1, \mu_2, \mu_3, \mu_4)$, we begin by writing

$$\begin{aligned} \xi_1\xi_2\xi_3\xi_4 = {} & (\xi_1 - \mu_1)(\xi_2 - \mu_2)(\xi_3 - \mu_3)(\xi_4 - \mu_4) + \mu_1(\xi_2 - \mu_2)(\xi_3 - \mu_3)(\xi_4 - \mu_4) \\ & + \mu_2(\xi_1 - \mu_1)(\xi_3 - \mu_3)(\xi_4 - \mu_4) + \mu_1\mu_2(\xi_3 - \mu_3)(\xi_4 - \mu_4) \\ & + \mu_3\xi_1\xi_2\xi_4 + \mu_4\xi_1\xi_2\xi_3 - \mu_3\mu_4\xi_1\xi_2 \end{aligned} \tag{9}$$

Before taking averages, first note that setting $N = 1$ in (5) yields

$$E[(\xi_1 - \mu_1)(\xi_2 - \mu_2)(\xi_3 - \mu_3)] = 0$$

or, upon rearranging,

$$E[\xi_1\xi_2\xi_3] = \mu_1E[\xi_2\xi_3] + \mu_2E[\xi_1\xi_3] + \mu_3E[\xi_1\xi_2] - 2\mu_1\mu_2\mu_3. \tag{10}$$

Now take the average of both sides of (9) over the joint Gaussian distribution centered at $\mu_1, \mu_2, \mu_3, \mu_4$. Making use of (4), (5), (10):

$$E[\xi_1\xi_2\xi_3\xi_4] = \sum\prod E[(\xi_i - \mu_i)(\xi_j - \mu_j)] + 0 + 0 + \mu_1\mu_2E[(\xi_3 - \mu_3)(\xi_4 - \mu_4)]$$

$$+\mu_3\{\mu_1 E[\xi_2\xi_4] + \mu_2 E[\xi_1\xi_4] + \mu_4 E[\xi_1\xi_2] - 2\mu_1\mu_2\mu_4\}$$
$$+\mu_4\{\mu_1 E[\xi_2\xi_3] + \mu_2 E[\xi_1\xi_3] + \mu_3 E[\xi_1\xi_2] - 2\mu_1\mu_2\mu_3\}$$
$$-\mu_3\mu_4 E[\xi_1\xi_2] \tag{11}$$

(Note that two terms with $E[\xi_1\xi_2]$ cancel.) Likewise, we can write

$$\begin{aligned}\xi_1\xi_2\xi_3\xi_4 &= (\xi_1+\mu_1)(\xi_2+\mu_2)(\xi_3+\mu_3)(\xi_4+\mu_4) - \mu_1(\xi_2+\mu_2)(\xi_3+\mu_3)(\xi_4+\mu_4)\\ &\quad - \mu_2(\xi_1+\mu_1)(\xi_3+\mu_3)(\xi_4+\mu_4) + \mu_1\mu_2(\xi_3+\mu_3)(\xi_4+\mu_4)\\ &\quad - \mu_3(\xi_1\xi_2\xi_4) - \mu_4\xi_1\xi_2\xi_3 - \mu_3\mu_4\xi_1\xi_2. \end{aligned}\tag{12}$$

Now take the average of both sides of (12) over the joint Gaussian distribution centered at $(-\mu_1, -\mu_2, -\mu_3, -\mu_4)$. Because this is a different average from (11), the expectation has an asterisk attached.

$$\begin{aligned}E[\xi_1\xi_2\xi_3\xi_4]^* &= \sum\prod E[(\xi_i+\mu_i)(\xi_j+\mu_j)]^* + \mu_1\mu_2 E[(\xi_3+\mu_3)(\xi_4+\mu_4)]^*\\ &\quad - \mu_3\{-\mu_1 E[\xi_2\xi_4]^* - \mu_2 E[\xi_1\xi_4]^* - \mu_4 E[\xi_1\xi_2]^* + 2\mu_1\mu_2\mu_4\}\\ &\quad - \mu_4\{-\mu_1 E[\xi_2\xi_3]^* - \mu_2 E[\xi_1\xi_3]^* - \mu_3 E[\xi_1\xi_2]^* + 2\mu_1\mu_2\mu_3\}\\ &\quad - \mu_3\mu_4 E[\xi_1\xi_2]^* \end{aligned}\tag{13}$$

But note that by symmetry

$$\sum\prod E[(\xi_i+\mu_i)(\xi_j+\mu_j)]^* = \sum\prod E[(\xi_i-\mu_i)(\xi_j-\mu_j)]$$

and

$$E[(\xi_3+\mu_3)(\xi_4+\mu_4)]^* = E[(\xi_3-\mu_3)(\xi_4-\mu_4)]$$

Also replacing ξ_i and ξ_j by $-\xi_i$ and $-\xi_j$, respectively, yields

$$E[\xi_i\xi_j]^* = E[\xi_i\xi_j]$$

The average of the product $x_1x_2x_3x_4$ over the *joint* mixed-Gaussian distribution is then obtained by summing the averages in (11), (13) and multiplying by $\frac{1}{2}$. Expanding $\sum\prod$ and collecting terms gives the result

$$\begin{aligned}E[x_1x_2x_3x_4] &= E[(\xi_1-\mu_1)(\xi_2-\mu_2)]E[(\xi_3-\mu_3)(\xi_4-\mu_4)]\\ &\quad + E[(\xi_1-\mu_1)(\xi_3-\mu_3)]E[(\xi_2-\mu_2)(\xi_4-\mu_4)]\\ &\quad + E[(\xi_1-\mu_1)(\xi_4-\mu_4)]E[(\xi_2-\mu_2)(\xi_3-\mu_3)]\\ &\quad + \mu_1\mu_2 E[(\xi_3-\mu_3)(\xi_4-\mu_4)]\\ &\quad + \mu_1\mu_3 E[\xi_2\xi_4] + \mu_2\mu_3 E[\xi_1\xi_4] + \mu_1\mu_4 E[\xi_2\xi_3]\\ &\quad + \mu_2\mu_4 E[\xi_1\xi_3] + \mu_3\mu_4 E[\xi_1\xi_2] - 4\mu_1\mu_2\mu_3\mu_4 \end{aligned}\tag{14}$$

The first group of terms on the right contains averages of products of pairs of joint Gaussian variables, e.g. $E[(\xi_1 - \mu_1)(\xi_2 - \mu_2)]$. This product is given by the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\xi_1 - \mu_1)(\xi_2 - \mu_2) f(\xi_1, \xi_2; \mu_1, \mu_2) d\xi_1 d\xi_2 \tag{15}$$

where $f(\xi_1, \xi_2; \mu_1, \mu_2)$ is the joint Gaussian distribution with parameters $\mu_1, \sigma_1;\ \mu_2, \sigma_2$. With a simple change of variables $\eta_1 = \xi_1 - \mu_1$, and $\eta_2 = \xi_2 - \mu_2$, the above integral becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \eta_1 \eta_2 f(\eta_1, \eta_2; 0, 0) d\eta_1 d\eta_2 \tag{16}$$

which is the usual expression for the average value of a product over the two-dimensional joint Gaussian distribution centered at (0, 0) (here $\rho = E[\eta_1 \eta_2]/\sigma_1 \sigma_2$).

The other terms in (14) are different. For example, the average $E[\xi_1 \xi_2]$ is given by the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \xi_1 \xi_2 f(\xi_1, \xi_2; \mu_1, \mu_2) d\xi_1 d\xi_2. \tag{17}$$

Using the same change of variables, this integral becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\eta_1 + \mu_1)(\eta_2 + \mu_2) f(\eta_1, \eta_2; 0, 0) d\eta_1 d\eta_2 \tag{18}$$

where the function $f(\cdot)$ is the same two-dimensional Gaussian distribution centered at (0, 0) as in (16), but now the average is w.r.t a different product. Expanding (18)

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\eta_1 + \mu_1)(\eta_2 + \mu_2) f(\eta_1, \eta_2; 0, 0) d\eta_1 d\eta_2 \\
&\quad = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \eta_1 \eta_2 f(\eta_1, \eta_2; 0, 0) d\eta_1 d\eta_2 \\
&\qquad + \mu_2 \int_{-\infty}^{\infty} \eta_1 \int_{-\infty}^{\infty} f(\eta_1, \eta_2; 0, 0) d\eta_2 d\eta_1 + \mu_1 \int_{-\infty}^{\infty} \eta_2 \int_{-\infty}^{\infty} f(\eta_1, \eta_2; 0, 0) d\eta_1 d\eta_2 \\
&\qquad + \mu_1 \mu_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\eta_1, \eta_2; 0, 0) d\eta_1 d\eta_2 \\
&\quad = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \eta_1 \eta_2 f(\eta_1, \eta_2; 0, 0) d\eta_1 d\eta_2 + \mu_1 \mu_2
\end{aligned} \tag{19}$$

as the inner integrals in the second and third terms are just the marginal distributions in η_1 and η_2 respectively and they have zero means. In general

$$E[(\xi_i-\mu_i)(\xi_j-\mu_j)]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\eta_i\eta_j f(\eta_i,\eta_j;0,0)d\eta_i d\eta_j=E[\eta_i\eta_j] \tag{20}$$

and

$$\begin{aligned}E[\xi_i\xi_j]&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\eta_i\eta_j f(\eta_i,\eta_j;0,0)d\eta_i d\eta_j+\mu_i\mu_j\\&=E[\eta_i\eta_j]+\mu_i\mu_j\end{aligned} \tag{21}$$

Therefore, (14) becomes:

$$\begin{aligned}E[x_1x_2x_3x_4]&=E[\eta_1\eta_2]E[\eta_3\eta_4]+E[\eta_1\eta_3]E[\eta_2\eta_4]+E[\eta_1\eta_4]E[\eta_2\eta_3]\\&\quad+\mu_1\mu_2E[\eta_3\eta_4]+\mu_1\mu_3(E[\eta_2\eta_4]+\mu_2\mu_4)+\mu_2\mu_3(E[\eta_1\eta_4]+\mu_1\mu_4)\\&\quad+\mu_1\mu_4(E[\eta_2\eta_3]+\mu_2\mu_3)\\&\quad+\mu_2\mu_4(E[\eta_1\eta_3]+\mu_1\mu_3)+\mu_3\mu_4(E[\eta_1\eta_2]+\mu_1\mu_2)-4\mu_1\mu_2\mu_3\mu_4\\&=E[\eta_1\eta_2]E[\eta_3\eta_4]+E[\eta_1\eta_3]E[\eta_2\eta_4]+E[\eta_1\eta_4]E[\eta_2\eta_3]\\&\quad+\mu_1\mu_2E[\eta_3\eta_4]+\mu_1\mu_3E[\eta_2\eta_4]+\mu_2\mu_3E[\eta_1\eta_4]+\mu_1\mu_4E[\eta_2\eta_3]\\&\quad+\mu_2\mu_4E[\eta_1\eta_3]+\mu_3\mu_4E[\eta_1\eta_2]+\mu_1\mu_2\mu_3\mu_4\end{aligned} \tag{22}$$

This is the desired result. Note that to calculate the average of the product of the four jointly mixed-Gaussian variables, only averages of products of pairs of zero-mean jointly Gaussian random variables need to be considered.

Finally, we note that, as might be expected, the average of a triple product for jointly mixed-Gaussian random variables is zero, i.e.,

$$\begin{aligned}E[x_1x_2x_3]&=\frac{1}{2}\times(\text{Average of } x_1x_2x_3 \text{ over joint Gaussian centered at } (\mu_1,\mu_2,\mu_3))\\&\quad+\frac{1}{2}\times(\text{Average of } x_1x_2x_3 \text{ over joint Gaussian centered at } (-\mu_1,-\mu_2,-\mu_3))\end{aligned} \tag{23}$$

which, from (10) becomes

$$\begin{aligned}E[x_1x_2x_3]&=\frac{1}{2}[\mu_1E[\xi_2\xi_3]+\mu_2E[\xi_1\xi_3]+\mu_3E[\xi_1\xi_2]-2\mu_1\mu_2\mu_3]\\&\quad+\frac{1}{2}[-\mu_1E[\xi_2\xi_3]-\mu_2E[\xi_1\xi_3]-\mu_3E[\xi_1\xi_2]-2(-\mu_1)(-\mu_2)(-\mu_3)]\\&=0\end{aligned} \tag{24}$$

□

3 Application: Auto-Bispectral Density Function

In the study of nonlinear random vibrations, higher-order spectral analysis has proven useful in a wide variety of applications ranging from detection to system identification. The higher-order spectra (HOS) are defined as the Fourier Transform of joint lagged cumulants. More specifically, given the collection of random variables $y(t_1), y(t_2), \ldots, y(t_n)$ and the associated joint cumulant $C(y(t_1), y(t_2), \ldots, y(t_n))$, the "$n$th" order cumulant spectrum is defined as [5]

$$S_{y^n}(\omega_1, \omega_2, \ldots, \omega_n) = \int_{\mathbb{R}^n} C(y(t_1), y(t_2), \ldots, y(t_n)) e^{-i(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n)} dt_1 dt_2 \cdots dt_n \tag{25}$$

In traditional spectral analysis the focus is on the power spectral density function ($n = 2$) where the assumption that the observed data are stationary reduces the number of frequency variables to one i.e. one computes $S_{yy}(\omega_1)$. Increasingly, however, both the bispectral density ($n = 3$) and trispectral density ($n = 4$) are being used in a number of applications. This increased use stems from the fact that the (severe) requirements for consistent estimators of these HOS can be easily met with modern computers (see [10] for a discussion of estimation of HOS).

The bispectrum is an appropriate tool for detecting the presence and magnitude of nonlinearity in a system given its time series response. Following (25) the bispectrum can be written

$$S_{yyy}(\omega_1, \omega_2, \omega_3) = \int_{\mathbb{R}^3} E[\tilde{y}(t_1)\tilde{y}(t_2)\tilde{y}(t_3)] e^{-i(\omega_1 t_1 + \omega_2 t_2 + \omega_3 t_3)} dt_1 dt_2 dt_3. \tag{26}$$

where, by definition, the arguments of the expected value are zero mean. For this reason we have adopted the notation $\tilde{y}(t) \equiv y(t) - \bar{y}(t)$ where $\bar{y}(t) = E[y(t)]$. If the data are assumed stationary, only the relative time lags between observations are important. Similarly, the signal means are no longer functions of time. Defining $\tau_1 = t_2 - t_1$ and $\tau_2 = t_3 - t_1$, and letting $t_1 \to t$ (26) becomes

$$\begin{aligned} S_{yyy}(\omega_2, \omega_3) &= \int_{\mathbb{R}^3} E[\tilde{y}(t)\tilde{y}(t+\tau_1)\tilde{y}(t+\tau_2)] e^{-i(\omega_2 \tau_1 + \omega_3 \tau_2)} e^{-i(\omega_1 + \omega_2 + \omega_3)t} d\tau_1 d\tau_2 dt \\ &= 2\pi\delta(\omega_1 + \omega_2 + \omega_3) \int_{\mathbb{R}^2} E[\tilde{y}(t)\tilde{y}(t+\tau_1)\tilde{y}(t+\tau_2)] e^{-i(\omega_2 \tau_1 + \omega_3 \tau_2)} d\tau_1 d\tau_2 \end{aligned} \tag{27}$$

where use has been made of the fact that the expected value does not depend on t. Thus, for stationary data the bispectral density only exists on the hyperplane $\omega_1 + \omega_2 + \omega_3 = 0$ and can be written in the more traditional form (letting $\omega_2 \to \omega_1$ and $\omega_3 \to \omega_2$)

$$S_{yyy}(\omega_1, \omega_2) = \int_{\mathbb{R}^2} E[\tilde{y}(t)\tilde{y}(t+\tau_1)\tilde{y}(t+\tau_2)] e^{-i(\omega_1 \tau_1 + \omega_2 \tau_2)} d\tau_1 d\tau_2 \tag{28}$$

An analytical solution for the bispectrum was recently developed in [7] for both single and multiple degree-of-freedom, quadratically nonlinear systems using a Volterra series approach. Similar expressions have also appeared in Brillinger [2, (3.9)], and in of Le Caillec and Garello [3, Appendix A (28)]. In these references the authors considered systems for which the input (driving) signal consisted of a stationary sequence of random variables taken from a Gaussian distribution. In this work the derivation was generalized to the case where the input signal $x(t)$ is again a stationary random process, distributed according to the more general mixed-Gaussian distribution given by (6). It will be shown that in the limit as $\mu \to 0$ one recovers the bispectrum derived in [7].

The derivation proceeds by substituting a Volterra series approximation of the signal $y(t)$ into (28) and simplifying. More specifically, assume

$$y(t) = \int_{\mathbb{R}} h_1(\tau)x(t-\tau)d\tau + \int_{\mathbb{R}^2} h_2(\tau_1,\tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2 \tag{29}$$

and take the expected value to give the signal mean

$$\begin{aligned}\bar{y} &= \int_{\mathbb{R}} h_1(\tau)E[x(t-\tau)]d\tau + \int_{\mathbb{R}^2} E[h_2(\tau_1,\tau_2)x(t-\tau_1)x(t-\tau_2)]d\tau_1 d\tau_2 \\ &= \int_{\mathbb{R}^2} h_2(\tau_1,\tau_2)E[x(t-\tau_1)x(t-\tau_2)]d\tau_1 d\tau_2,\end{aligned} \tag{30}$$

provided that the input signal $x(t)$ has zero mean. Substitution of (29), (30) into the expectation in (28) yields 8 separate terms. However, 4 of these terms contain expectations of products of odd numbers of the input $x(t)$ and are therefore, by (24), zero. Another of the four remaining terms results only in higher-order contributions to the bispectrum (as was shown in [7]) and is therefore not considered here. The expected value required of (28) is therefore given by

$$\begin{aligned}&E[\tilde{y}(t)\tilde{y}(t+\tau_1)\tilde{y}(t+\tau_2)] \\ &= E\Big[\int_{\mathbb{R}} h_1(\tau_3)x(t+\tau_1-\tau_3)d\tau_3 \int_{\mathbb{R}} h_1(\tau_4)x(t+\tau_2-\tau_4)d\tau_4 \int_{\mathbb{R}^2} h_2(\tau_5,\tau_6) \\ &\quad\times \big(x(t-\tau_5)x(t-\tau_6) - E[x(t-\tau_5)x(t-\tau_6)]\big)d\tau_5 d\tau_6 \\ &\quad+ \int_{\mathbb{R}} h_1(\tau_3)x(t-\tau_3)d\tau_3 \int_{\mathbb{R}} h_1(\tau_4)x(t+\tau_2-\tau_4)d\tau_4 \int_{\mathbb{R}^2} h_2(\tau_5,\tau_6) \\ &\quad\times \big(x(t+\tau_1-\tau_5)x(t+\tau_1-\tau_6) - E[x(t+\tau_1-\tau_5)x(t+\tau_1-\tau_6)]\big)d\tau_5 d\tau_6 \\ &\quad+ \int_{\mathbb{R}} h_1(\tau_3)x(t-\tau_3)d\tau_3 \int_{\mathbb{R}} h_1(\tau_4)x(t+\tau_1-\tau_4)d\tau_4 \int_{\mathbb{R}^2} h_2(\tau_5,\tau_6) \\ &\quad\times \big(x(t+\tau_2-\tau_5)x(t+\tau_2-\tau_6) - E[x(t+\tau_2-\tau_5)x(t+\tau_2-\tau_6)]\big)d\tau_5 d\tau_6\Big]\end{aligned} \tag{31}$$

These three terms are denoted $E_I[\cdot]$, $E_{II}[\cdot]$, $E_{III}[\cdot]$ respectively. The first of these terms is given by

$$\begin{aligned}&E_I[\tilde{y}(t)\tilde{y}(t+\tau_1)\tilde{y}(t+\tau_2)] \\ &\quad= \int_{\mathbb{R}^4} h_1(\tau_3)h_1(\tau_4)h_2(\tau_5,\tau_6)E\big[x(t+\tau_1-\tau_3)x(t+\tau_2-\tau_4) \\ &\qquad\times \big(x(t-\tau_5)x(t-\tau_6) - E[x(t-\tau_5)x(t-\tau_6)]\big)\big]d\tau_{3\to 6}.\end{aligned} \tag{32}$$

For notational convenience expectations of products of two random variables are defined as

$$
\begin{aligned}
E[x(t-\tau_a)x(t-\tau_b)] &\equiv \phi_{xx}^{a,b} \\
E[x(t+\tau_a-\tau_b)x(t-\tau_c)] &\equiv \phi_{xx}^{a,b,c} \\
E[x(t+\tau_a-\tau_b)x(t+\tau_c-\tau_d)] &\equiv \phi_{xx}^{a,b,c,d}
\end{aligned} \tag{33}
$$

which is simply the cross-correlation function. It will also prove useful for this derivation to write these auto-correlations in terms of the associated power spectral density function. By the Weiner-Khintchine relationship, (33) are related to their frequency domain counterparts by Fourier Transform as

$$
\begin{aligned}
\phi_{xx}^{a,b} &= \frac{1}{2\pi}\int_{\mathbb{R}} S_{xx}(\omega)e^{i\omega(-\tau_b+\tau_a)}d\omega \\
\phi_{xx}^{a,b,c} &= \frac{1}{2\pi}\int_{\mathbb{R}} S_{xx}(\omega)e^{i\omega(-\tau_c-\tau_a+\tau_b)}d\omega \\
\phi_{xx}^{a,b,c,d} &= \frac{1}{2\pi}\int_{\mathbb{R}} S_{xx}(\omega)e^{i\omega(\tau_c-\tau_d-\tau_a+\tau_b)}d\omega
\end{aligned} \tag{34}
$$

Returning to (32)

$$
\begin{aligned}
&E_I[\tilde{y}(t)\tilde{y}(t+\tau_1)\tilde{y}(t+\tau_2)] \\
&\quad = \int_{\mathbb{R}^4} h_1(\tau_3)h_1(\tau_4)h_2(\tau_5,\tau_6)E[x(t+\tau_1-\tau_3)x(t+\tau_2-\tau_4) \\
&\qquad \times (x(t-\tau_5)x(t-\tau_6)-\phi_{xx}^{5,6})]d\tau_{3\to 6}.
\end{aligned} \tag{35}
$$

we are left with an expectation of the product of four random variables that are jointly mixed-Gaussian distributed. At this point we make use of Theorem 2.1 and reduce this product to a summation of second order products of Gaussian distributed random variables as

$$
\begin{aligned}
&E_I[\tilde{y}(t)\tilde{y}(t+\tau_1)\tilde{y}(t+\tau_2)] \\
&\quad = \int_{\mathbb{R}^4} h_1(\tau_3)h_1(\tau_4)h_2(\tau_5,\tau_6)\big\{\phi_{xx}^{1,3,2,4}\phi_{xx}^{5,6}+\phi_{xx}^{1,3,5}\phi_{xx}^{2,4,6}+\phi_{xx}^{1,3,6}\phi_{xx}^{2,4,5} \\
&\qquad + \mu^2(\phi_{xx}^{5,6}+\phi_{xx}^{2,4,6}+\phi_{xx}^{1,3,6}+\phi_{xx}^{2,4,5}+\phi_{xx}^{1,3,5}+\phi_{xx}^{1,3,2,4}) \\
&\qquad - \phi_{xx}^{1,3,2,4}\phi_{xx}^{5,6}+\mu^4\big\}d\tau_{3\to 6}
\end{aligned} \tag{36}
$$

We also note that by symmetry of the second-order Volterra kernel, $h_2(\tau_5,\tau_6)=h_2(\tau_6,\tau_5)$, some of the above terms can be combined such that the expression to be simplified is

$$
\begin{aligned}
&E_I[\tilde{y}(t)\tilde{y}(t+\tau_1)\tilde{y}(t+\tau_2)] \\
&\quad = \int_{\mathbb{R}^4} h_1(\tau_3)h_1(\tau_4)h_2(\tau_5,\tau_6)\big\{2\phi_{xx}^{1,3,5}\phi_{xx}^{2,4,6} \\
&\qquad + \mu^2(\phi_x x^{5,6}+2\phi_{xx}^{2,4,6}+2\phi_{xx}^{1,3,6}+\phi_{xx}^{1,3,2,4})+\mu^4\big\}d\tau_{3\to 6}.
\end{aligned} \tag{37}
$$

Each of the six integrals in (37) may be simplified using the same general procedure. Simplification of this first integral, denoted Ia, was performed in [7] and is repeated here for completeness.

$$
\begin{aligned}
&E_{Ia}[\tilde{y}(t)\tilde{y}(t+\tau_1)\tilde{y}(t+\tau_2)]\\
&= \int_{\mathbb{R}^4} 2h_1(\tau_3)h_1(\tau_4)h_2(\tau_5,\tau_6)\phi_{xx}^{1,3,5}\phi_{xx}^{2,4,6}d\tau_{3\to 6}\\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^6} 2h_1(\tau_3)h_1(\tau_4)h_2(\tau_5,\tau_6)S_{xx}(\omega_1)e^{i\omega_1(-\tau_5-\tau_1+\tau_3)}S_{xx}(\omega_2)\\
&\quad\times e^{i\omega_2(-\tau_6+\tau_4-\tau_2)}d\tau_{3\to 6}d\omega_1 d\omega_2\\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} 2\int_{\mathbb{R}} h_1(\tau_3)e^{i\omega_1\tau_3}d\tau_3 \int_{\mathbb{R}} h_1(\tau_4)e^{i\omega_2\tau_4}d\tau_4\\
&\quad\times \int_{\mathbb{R}^2} h_2(\tau_5,\tau_6)e^{-i(\omega_1\tau_5+\omega_2\tau_6)}d\tau_5 d\tau_6 S_{xx}(\omega_1)S_{xx}(\omega_2)e^{-i(\omega_1\tau_1+\omega_2\tau_2)}d\omega_1 d\omega_2\\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} 2H_1(-\omega_1)H_1(-\omega_2)H_2(\omega_1,\omega_2)S_{xx}(\omega_1)S_{xx}(\omega_2)e^{-i(\omega_1\tau_1+\omega_2\tau_2)}d\omega_1 d\omega_2\\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} 2H_1(\omega_1)H_1(\omega_2)H_2(-\omega_1,-\omega_2)S_{xx}(\omega_1)S_{xx}(\omega_2)e^{i(\omega_1\tau_1+\omega_2\tau_2)}d\omega_1 d\omega_2 \qquad (38)
\end{aligned}
$$

where use has been made of the fact that the auto-spectral density function is symmetric i.e. $S_{xx}(-\omega) = S_{xx}(\omega)$. Expression (38) can immediately be recognized as an inverse double Fourier Transform with respect to the delays τ_1, τ_2. Thus, by (28), the argument of this double integral is the desired component of the bispectral density, i.e.

$$
B_{Ia}(\omega_1,\omega_2) = 2H_1(\omega_1)H_1(\omega_2)H_2(-\omega_1,-\omega_2)S_{xx}(\omega_1)S_{xx}(\omega_2) \qquad (39)
$$

The same procedure can be applied to each of the other integrals, however these terms only contribute to the bispectrum along various submanifolds of the full ω_1, ω_2 plane. For example, the third integral is simplified as

$$
\begin{aligned}
&E_{Ic}[\tilde{y}(t)\tilde{y}(t+\tau_1)\tilde{y}(t+\tau_2)]\\
&= \int_{\mathbb{R}^4} 2\mu^2 h_1(\tau_3)h_1(\tau_4)h_2(\tau_5,\tau_6)\phi_{xx}^{2,4,6}d\tau_{3\to 6}\\
&= \frac{1}{2\pi}\int_{\mathbb{R}^5} 2\mu^2 h_1(\tau_3)h_1(\tau_4)h_2(\tau_5,\tau_6)S_{xx}(\omega_2)e^{i\omega_2(-\tau_6+\tau_4-\tau_2)}d\tau_{3\to 6}d\omega_2\\
&= \frac{1}{2\pi}\int_{\mathbb{R}} 2\mu^2 H_1(0)H_1(-\omega_2)H_2(0,\omega_2)S_{xx}(\omega_2)e^{-i\omega_2\tau_2}d\omega_2\\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} 4\pi\mu^2 H_1(0)H_1(\omega_2)H_2(0,-\omega_2)S_{xx}(-\omega_2)\delta(\omega_1)e^{i(\omega_1\tau_1+\omega_2\tau_2)}d\omega_1 d\omega_2 \qquad (40)
\end{aligned}
$$

where the $\delta(\omega_1)$ has been added in order to force (40) into the form of an inverse double Fourier transform. Thus the associated component of the bispectrum is

$$
B_{Ic}(\omega_1,\omega_2) = 4\pi\mu^2 H_1(0)H_1(\omega_2)H_2(0,-\omega_2)S_{xx}(-\omega_2)\delta(\omega_1). \qquad (41)
$$

This same procedure can be followed for each of the remaining terms in the expectation $E_I[\cdot]$. Similarly, the other two expected values $E_{II}[\cdot], E_{III}[\cdot]$ can be simplified to yield the final expression for the bispectral density.

$$
\begin{aligned}
B(\omega_1,\omega_2) = {} & 2H_1(\omega_1)H_1(\omega_2)H_2(-\omega_1,-\omega_2)S_{xx}(-\omega_1)S_{xx}(-\omega_2) \\
& + 2H_1(-\omega_1-\omega_2)H_1(\omega_2)H_2(\omega_1+\omega_2,-\omega_2)S_{xx}(\omega_1+\omega_2)S_{xx}(-\omega_2) \\
& + 2H_1(-\omega_1-\omega_2)H_1(\omega_1)H_2(\omega_1+\omega_2,-\omega_1)S_{xx}(\omega_1+\omega_2)S_{xx}(-\omega_1) \\
& + 6\pi\mu^2 H_1(0)^2 \int_{\mathbb{R}} H_2(-\omega,\omega)S_{xx}(\omega)d\omega\delta(\omega_1)\delta(\omega_2) \\
& + 2\pi\mu^2[H_1(\omega_1)H_1(\omega_2)H_2(0,0)S_{xx}(\omega_2) + 2H_1(0)H_1(\omega_2)H_2(0,\omega_1)S_{xx}(\omega_1) \\
& + 2H_1(0)H_1(\omega_1)H_2(0,\omega_2)S_{xx}(\omega_2)]\delta(\omega_1+\omega_2) \\
& + 2\pi\mu^2[2H_1(0)H_1(\omega_2)H_2(0,-\omega_2)S_{xx}(\omega_2) + |H_1(\omega_2)|^2H_2(0,0)S_{xx}(\omega_2) \\
& + 2H_1(-\omega_2)H_1(0)H_2(0,\omega_2)S_{xx}(\omega_2)]\delta(\omega_1) \\
& + 2\pi\mu^2[2H_1(0)H_1(\omega_1)H_2(0,-\omega_1)S_{xx}(\omega_1) + |H_1(\omega_1)|^2H_2(0,0)S_{xx}(\omega_1) \\
& + 2H_1(-\omega_1)H_1(0)H_2(0,\omega_1)S_{xx}(\omega_1)]\delta(\omega_2) \\
& + 12\pi^2\mu^4 H_1(0)H_1(0)H_2(0,0)\delta(\omega_1)\delta(\omega_2)
\end{aligned} \tag{42}
$$

For zero mean Gaussian inputs ($\mu = 0$) the above expression reduces to the same one derived in [7]. However, this more general formulation allows for the possibility of strongly non-Gaussian inputs $\mu > 0$. As an example of how the above expression can be used, consider the single-degree-of-freedom, quadratically nonlinear system

$$m\ddot{y}(t) + c_1\dot{y}(t) + c_2\dot{y}^2(t) + k_1 y(t) + k_2 y^2(t) = Ax(t) \tag{43}$$

where m, c_1, c_2, k_1, k_2, A are constant coefficients. The Volterra kernels H_1 and H_2 associated with this system may be obtained via the harmonic probing approach outlined in Worden & Tomlinson [19]. Taking the displacement $y(t)$ as the response variable of interest, it can be easily shown that

$$
\begin{aligned}
H_1(\omega) &= \frac{1}{k_1 + ic_1\omega - m\omega^2} \\
H_2(\omega_1,\omega_2) &= (-k_2 + c_2\omega_1\omega_2)H_1(\omega_1)H_1(\omega_2)H_1(\omega_1+\omega_2).
\end{aligned} \tag{44}
$$

The strength of the quadratic nonlinearities in both dissipative and restoring terms are governed by c_2 and k_2 respectively. The input $x(t)$ is taken to consist of iid realizations of a mixed-Gaussian random process. In what follows the linear parameters are fixed to the values $m = 1$ [kg], $k_1 = 1000$ [N/m], $c_1 = 3.0$ [N s/m]. The amplitude A depends on the power spectral density $S_{xx}(\omega)$ appearing in the theoretical auto-bispectrum (42). Under the assumption of iid input, the input spectrum is a constant given by $S_{xx}(\omega) = P$. For this work we have used $P = 0.01$ [N^2/Hz] throughout.

Figure 3 shows a plot of the auto-bispectral density for $\mu = 0$ (Gaussian input) and $\mu = 1$ using the parameters $k_2 = 100000$ [$\mathrm{N/m^2}$], $c_2 = 0$ [$\mathrm{N\,s^2/m^2}$]. As the input begins to deviate from a pure Gaussian distribution, the resulting bispectral density begins to show increased peak values along the lines $\omega_1 = 2\pi f_1 = 0$, $\omega_2 = 2\pi f_2 = 0$, $f_1 = -f_2$ and at the origin. Further increases in μ result in further increases in the peaks. This result holds implications for nonlinearity detection where bispectral peaks are frequently used detection statistics (see e.g. [8, 18]). Simply changing the *form* of the input from iid Gaussian noise to iid Mixed-Gaussian noise can result in significant increases in the heights of the bispectral peaks, despite the input power P remaining constant.

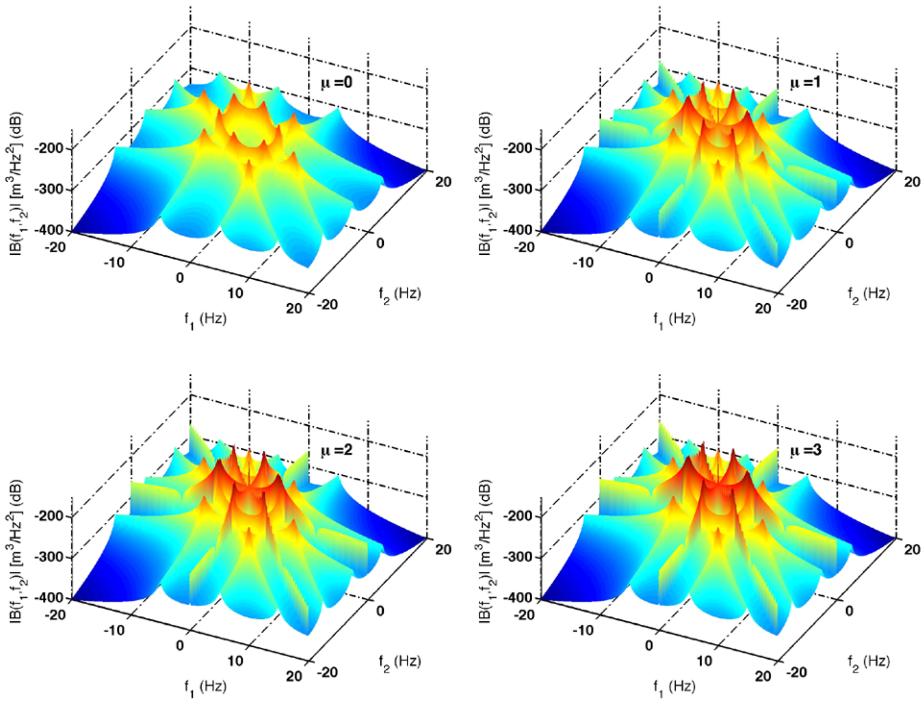


Fig. 3 Magnitude auto-bispectral density obtained for Gaussian input $\mu = 0$ and for increasingly non-Gaussian input $\mu = 1, 2, 3$. Results are plotted for a stiffness nonlinearity of $k_2 = 10^5$ N/m^2, with $c_2 = 0$ N s^2/m^2

One way to better understand this result is to think back to the more familiar second order cumulant spectrum, the power spectral density function $S_{xx}(\omega)$. For a signal $x(t)$ with non-zero mean $\bar{x}$, we have $S_{xx}(0) \sim \bar{x}^2$ or, alternatively, $S_{xx}(\omega)\delta(\omega) \sim \bar{x}^2$. In other words, the non-zero mean leads to a term in the zeroth frequency "bin". Similarly, in (42) the system nonlinearity combined with the non-zero offset μ of the input distribution leads to an increasing non-zero mean in the response. The result is increased values of the bispectrum along certain frequency *lines* e.g. $\omega_1 = 0$ as opposed to the zeroth frequency bin as for the power spectrum.

It is worth mentioning that the bispectra plotted in Fig. 3 correspond to the displacement output of the system (43), i.e. the signal of interest was $y(t)$ [m]. However, expression (42) could have been used to look at the bispectrum associated with $\dot{y}(t)$ [m/s] or $\ddot{y}(t)$ [m/s^2]. Interestingly these bispectra turn out to be identical to those obtained for pure Gaussian forcing. The reason concerns the terms in (42) evaluated at $\omega = 0$ i.e., $H_1(0)$, $H_2(0, 0)$. The Volterra kernels associated with either velocity or acceleration are of the same form as (44), however they are multiplied by the frequency variables. For velocity and acceleration, $H_1(\omega)$ is multiplied by $i\omega$ and $-\omega^2$ respectively. Similarly, for $H_2(\omega_1, \omega_2)$ the multiplier is $i(\omega_1 + \omega_2)$ and $-(\omega_1 + \omega_2)^2$ respectively. Thus, (42) associated with the velocity or acceleration output of the system governed by (43) will be the same, regardless of whether the input forcing distribution is Gaussian or mixed-Gaussian. Only the displacement bispectrum changes for mixed Gaussian forcing.

4 Conclusions

In this work we have derived a version of Isserlis' Theorem (Wick's Theorem) to handle the case of four jointly mixed-Gaussian, random variables. This theorem was then used to derive an expression for the auto-bispectral density of a general second-order, quadratically nonlinear system subject to a mixed-Gaussian input.

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